

# Resonant magnetic perturbations and divertor footprints in poloidally diverted tokamaks

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General formula describing both the divertor strike point splitting and width of magnetic islands created by resonant magnetic perturbations (RMPs) in a poloidally diverted tokamak equilibrium is derived. Under the assumption that the RMP is produced by coils at the low-field side such as those used to control edge localized modes (ELMs) it is demonstrated that the width of islands on different magnetic surfaces at the edge and the amount of divertor splitting are related to each other. Explanation is provided of aligned maxima of the perturbation spectra with the safety factor profile – an effect empirically observed in models of many perturbation coil designs.

## I. INTRODUCTION

Resonant magnetic perturbations (RMPs) are being investigated as tool to control edge localized modes (ELMs), in particular their application to ITER is foreseen. The RMPs for ELM control are produced by coils whose design is specific to each tokamak. This method is reminiscent of the ergodic divertor on tokamaks Tore Supra and TEXTOR which also essentially relies on RMPs produced by dedicated systems of coils, and many aspects are similar, especially the formation of magnetic islands on rational surfaces and possible stochastic transport when the islands overlap. However the presence of X-point in the poloidally diverted tokamaks provides some unique features: the splitting of divertor strike points, or divertor footprints – a signature of the homoclinic tangle created by the perturbation<sup>1</sup>, and the divergence of the safety factor profile at the separatrix, due to which the number of rational surfaces even for one toroidal mode is infinite and island overlap on them is facilitated. The noncircular geometry of the plasma cross-section also complicates analytical treatment of magnetic islands, requiring cautious use of non-orthogonal coordinate systems.

Since both the homoclinic tangle and magnetic islands with the resulting stochasticity are consequences of the perturbation, it is natural to ask if there is any relation between them. It has been known that while every island chain is related to a different poloidal mode of the perturbation, their sizes can be expressed by a single function – the Poincaré-type integral which also generalizes to an expression for the length of the divertor footprints<sup>2</sup>. The present paper uses instead the more familiar formalism of the Melnikov integral<sup>3</sup> (a standard tool for the analysis of the homoclinic tangle) and explores the radial dependence of this function in the often encountered case of RMPs from the perturbation coils localized at the low-field side (LFS), as it is the case for most ELM control coil designs.

The paper starts with a review of the method to calculate island sizes. Care is taken to provide a formula valid in a plasma with a general aspect ratio and non-circular

cross-section as this is essential for the edge pedestal region near the separatrix which is crucial for ELM control. It is shown how the island sizes are determined by a function whose definition is not affected by the coordinate singularity on the separatrix, and which we call the Melnikov-like function. The homoclinic tangle and divertor footprints are then explained together with the method of Melnikov integral for determining analytically the divertor footprints length. A particularly simple expression is given for the case when the perturbation has only one dominant toroidal mode. The relation between Melnikov integral and the Melnikov-like function is explained. Then we restrict our treatment to the case of perturbation localized on the LFS. Under this assumption the relation between modes at the different surfaces is derived and it is shown how does the divertor footprint length relate to the sizes of magnetic islands at the edge.

## II. WIDTH OF MAGNETIC ISLANDS IN A REALISTIC GEOMETRY

### A. Expression using the resonant modes of the perturbation

Nonaxisymmetric magnetic perturbations of a tokamak magnetic field produce chains of magnetic islands on magnetic surfaces with low-order rational values of the safety factor  $q$ . Those islands are created by resonances of the perturbation field  $\delta\mathbf{B}$  with the unperturbed field lines on the rational surface. To express these resonances we use a magnetic coordinate system<sup>4</sup>  $(s, \theta^*, \varphi)$  where  $s$  is a flux surface label,  $\theta^*$  the poloidal coordinate and  $\varphi$  the toroidal coordinate. The angular coordinates  $\theta^*, \varphi$  are chosen as in the PEST coordinates<sup>5</sup> and the radial coordinate  $s$  is defined as square root of the normalized poloidal magnetic flux  $\psi_N$ :  $s = \sqrt{\psi_N}$ , in accordance with previous works (e.g. Refs. 6,7). The coordinate  $\varphi$  is simply the geometric toroidal angle and the coordinate system has the property that  $\theta^* = o\varphi/q(s) + \text{const.}$  along a field line, where  $o$  represents the orientation of the magnetic field:  $o = 1$  for the left-handed field and

$o = -1$  for the right-handed field<sup>17</sup>. The islands are created by the  $(m, n)$  Fourier component  $\tilde{b}_{(m,n)}^1$  of the normalized perturbation  $\delta b^1 \equiv B^1/B^3$ , where  $B^1 \equiv \delta \vec{B} \cdot \nabla s$  is the contravariant  $s$  component of the perturbation and  $B^3 \equiv \vec{B} \cdot \nabla \varphi$  is the contravariant  $\varphi$  component of the equilibrium field. The Fourier transform is taken with respect to the  $\theta^*$  and  $\varphi$  coordinates, thus we have

$$\delta b^1 = \sum_{m,n=-\infty}^{\infty} \tilde{b}_{(m,n)}^1 \exp[i(m\theta^* - n\varphi)] \quad (1)$$

and the Fourier harmonics can be obtained as

$$\tilde{b}_{(m,n)}^1 = \frac{1}{(2\pi)^2} \oint \exp[-i(m\theta^* - n\varphi)] \delta b^1 d\theta^* d\varphi. \quad (2)$$

A Fourier component is resonant with the unperturbed field lines when  $q = om/n$ . The values  $\tilde{b}_{(m,n)}^1$  are complex, and as  $\delta b^1$  is real, the following relation for the complex conjugate holds:  $(\tilde{b}_{(m,n)}^1)^* = \tilde{b}_{(-m,-n)}^1$ . An alternative to (1) is a representation using purely real coefficients:

$$\delta b^1 = \sum_{m=-\infty, n=1}^{\infty} b_{(m,n)}^1 \sin(m\theta^* - n\varphi + \chi_{mn}) \quad (3)$$

$$b_{(m,n)}^1 = 2|\tilde{b}_{(m,n)}^1| \quad (4)$$

$$\chi_{mn} = \arg \tilde{b}_{(m,n)}^1 \quad (5)$$

Widely used formulae exist for determining the width of magnetic islands from the Fourier spectrum of the perturbation<sup>6,8</sup>. They are typically derived in a cylindrical geometry where the toroidal curvature is not being taken into account (the toroidal magnetic field is considered constant), thus  $B^3$  in the expression for  $\delta b^1$  is approximated by its value at the magnetic axis. As noted in Ref 9, this leads to an error in estimating the island width. In the example of TEXTOR and its dynamic ergodic divertor (DED), the island size was overestimated because the DED coils are located at the high-field side, thus the actual value of  $B^3$  is larger and  $\delta b^1$  is smaller than in the cylindrical approximation. For the ergodic divertor of Tore Supra, which was located at the low-field side, the island sizes were underestimated. It should be noted that while the toroidal field magnitude  $B_T$  varies with the radial distance  $R$  from the major axis as  $B_T \propto 1/R$ , for the contravariant component the dependence is stronger:  $B^3 \propto 1/R^2$ .

Moreover the cylindrical formula for island widths uses  $r$  (the distance from the magnetic axis) as a radial coordinate and thus is valid only in a situation where the magnetic surfaces have circular and concentric cross-sections. In divertor tokamaks we are far from this geometry, especially in the edge region near the separatrix which is the most important when perturbations are used as an ELM control mechanism. We thus need a formula which would be usable in a general geometry, with a varying toroidal

field and noncircular flux surfaces, using for example the coordinate  $s$  as a general flux surface label instead of  $r$ .

To derive this formula we introduce new coordinates  $\chi \equiv \theta^* - n/m\varphi$  and  $\bar{s} \equiv s - s_0$  where  $s_0$  is the flux label of the resonant surface where  $q = m/n$ . The differential equation of the field line are

$$\frac{ds}{d\varphi} = \frac{B^1}{B^3} \quad (6)$$

$$\frac{d\theta^*}{d\varphi} = \frac{1}{q} \quad (7)$$

Using the coordinates  $\chi$  and  $\bar{s}$  and the relation

$$\frac{d\chi}{d\varphi} = \frac{1}{q} - \frac{n}{m} \quad (8)$$

the equation (6) becomes

$$1/q - n/m \, d\bar{s} = B^1/B^3 \, d\chi \quad (9)$$

Keeping only the resonant part of the perturbation, thus substituting  $B^1/B^3$  by  $b_{(m,n)}^1 \sin(m\chi + \chi_{mn})$ , we obtain

$$1/q - n/m \, d\bar{s} = b_{(m,n)}^1 \sin(m\chi + \chi_{mn}) \, d\chi \quad (10)$$

Using a linear approximation of the left side, we obtain

$$\left. \frac{dq^{-1}}{d\bar{s}} \right|_{q=m/n} \bar{s} \, d\bar{s} \approx b_{(m,n)}^1 \sin(m\chi + \chi_{mn}) \, d\chi \quad (11)$$

This equation can be easily integrated to obtain an algebraic equation for field lines:

$$\bar{s}^2 \approx \frac{2q^2 b_{(m,n)}^1}{q'm} [\cos(m\chi + \chi_{mn}) + C] \quad (12)$$

where  $q' \equiv dq/ds$  at the resonant surface and  $C$  is an integration constant. The choice  $C = 1$  corresponds to the island separatrix whose maximum radial excursion is the island half-width  $\delta$ , given by the formula

$$\delta = \sqrt{\frac{4q^2 b_{(m,n)}^1}{q'm}} \quad (13)$$

An alternative to this approach is to use a Hamiltonian approach where the field lines are interpreted as trajectories of a Hamiltonian dynamical system whose Hamilton function is the poloidal flux and the perturbation is represented as a perturbed Hamiltonian (flux). This approach has been used in many theoretical works. We briefly review it in the appendix A and prove its equivalence to the approach described above. It should be emphasized that the hamiltonian approach automatically includes correctly the effects of toroidal geometry and non-circular cross-section – no corrections are necessary. It is however still important to have a correct formula using the perturbed magnetic field (13), because this is the approach usually used in numerical studies of perturbation coil designs, as the perturbed field can be readily calculated from the coil geometry by the Biot-Savart formula. We will also see that the harmonics of the perturbed field are directly related to the Melnikov function.

## B. Expression using Melnikov-type integral

The coordinate system  $(\theta^*, s)$  on a poloidal plane has a singularity at the separatrix. It is useful to define a value characterizing magnetic islands which, unlike  $\tilde{b}_{(m,n)}^1$ , will not use the  $\theta^*$  coordinate, so it will stay well-defined even at the separatrix.

We start by defining a coordinate  $\phi$  which will be used instead of  $\theta^*$ . We follow a procedure used in the definition of the separatrix map and the Melnikov integral<sup>3,10</sup>. For every magnetic surface the point on the outboard midplane has  $\phi = 0$ . Following a field line parameterized by the toroidal angle  $\varphi$  from this point, we assign to any other point on the field line the value  $\phi = \varphi$ . Thus  $\phi$  of a given point is the toroidal angle needed to reach it by following a field line from the outboard midplane. Since the field line returns to the same poloidal position after making  $q$  toroidal turns, the range of the coordinate  $\phi$  needed to cover a magnetic surface in the poloidal plane is  $(-q\pi, q\pi)$  where the endpoints of this interval are identified with each other. Together with a flux surface label such as  $s$  we obtain a coordinate system on the poloidal plane. The separatrix is a special case: it is covered by  $\phi \in (-\infty, \infty)$  since  $q$  is infinite on the separatrix, and the X-point corresponds to  $\phi = \pm\infty$ . In this case  $\phi$  is called a *homoclinic coordinate*<sup>3</sup>. As on a field line  $\varphi = oq\theta^* + \text{const.}$  and  $\theta^* = 0$  on the outboard midplane where  $\phi = 0$ , the relation between  $\theta^*$  and  $\phi$  is  $\phi = oq\theta^*$ . Using this relation, the definition (2) of  $\tilde{b}_{(m,n)}^1$  can be rewritten as

$$\tilde{b}_{(m,n)}^1 = \frac{1}{oq(2\pi)^2} \int_{-q\pi}^{q\pi} \int_0^{2\pi} \exp \left[ -i \left( o \frac{m}{q} \phi - n\varphi \right) \right] \delta b^1 d\varphi d\phi \quad (14)$$

If we define a toroidal perturbation Fourier mode  $\tilde{b}_n^1$  as

$$\tilde{b}_n^1(\phi) = \frac{1}{2\pi} \int_0^{2\pi} \exp(in\varphi) \delta b^1(\phi, \varphi) d\varphi \quad (15)$$

we may write (14) as

$$\tilde{b}_{(m,n)}^1 = \frac{1}{o2\pi q} \int_{-q\pi}^{q\pi} \exp \left[ -io \left( \frac{m}{q} \phi \right) \right] \tilde{b}_n^1(\phi) d\phi \quad (16)$$

On resonant surfaces with  $q = om/n$  this may be simplified to

$$\tilde{b}_{(m=onq,n)}^1 = \frac{1}{o2\pi q} \int_{-q\pi}^{q\pi} \exp(-in\phi) \tilde{b}_n^1(\phi) d\phi \quad (17)$$

( $\tilde{b}_n^1$ ,  $\tilde{b}_{(m,n)}^1$ ) and  $\delta b^1$  all depend also on the magnetic surface, this was omitted from the expressions above for brevity. In (17) there appears a complex Melnikov-like function  $\tilde{S}_n(s)$  given by

$$\tilde{S}_n(s) \equiv \int_{-q\pi}^{q\pi} \exp(-in\phi) \tilde{b}_n^1(s, \phi) d\phi, \quad (18)$$

defined using the coordinate  $\phi$  which does not have a singularity at the separatrix, so the definition can be extended to the separatrix:

$$\tilde{S}_n(s=1) \equiv \int_{-\infty}^{\infty} \exp(-in\phi) \tilde{b}_n^1(s=1, \phi) d\phi. \quad (19)$$

The function  $\tilde{S}_n$  fulfills our requirement: it can replace  $\tilde{b}_{(m,n)}^1 = \tilde{S}_n/(o2\pi q)$  and is defined using values which remain regular at the separatrix. The island width (13) can be expressed using  $\tilde{S}_n$  instead of  $\tilde{b}_{(m,n)}^1$ :

$$\delta = \sqrt{\frac{4|\tilde{S}_n|}{n\pi q'}}. \quad (20)$$

The only remaining divergent term in (20) is the shear  $q'$  which grows to infinity at the separatrix. This dependence is physical: its consequence is that island width has a zero limit at the separatrix.

## III. DIVERTOR FOOTPRINTS

Since the particle and heat transport are mostly parallel to field lines, the patterns of particle and heat flux to the divertor plates can be expected to be related to the divertor magnetic footprints, i.e. the patterns of intersections of field lines with the divertor. Field lines which carry heat and particle fluxes from inside the plasma are those with a high connection length, i.e. the number of toroidal turns following the field line in the plasma before it reaches the wall again.

Since the field lines can be interpreted as trajectories of a Hamiltonian dynamical system with the toroidal angle in the role of the time, methods of the theory of Hamiltonian systems can be used. A concept especially useful for the study of divertor footprints is the one of invariant manifolds<sup>1</sup>. An invariant manifold is a surface in the phase space of the dynamical system which remains invariant by the time evolution of the system, thus a trajectory with an initial point on the invariant manifold is constrained to remain on it<sup>3</sup>. In our case the trajectories are field lines and one example of invariant manifolds are the magnetic surfaces of the toroidally symmetric tokamak equilibrium. A particularly interesting case of invariant manifolds are the stable and unstable manifolds of hyperbolic fixed points. A stable manifold is formed by field lines asymptotically approaching the fixed point, while the unstable manifold is formed by field lines asymptotically leaving the fixed point. The definition depends on the direction in which the field lines are followed. If we follow them in the opposite direction, the stable manifold becomes unstable and vice versa. In plasma equilibria the hyperbolic fixed points are called X-points and are associated with the poloidal divertor or with magnetic islands. An example of invariant manifolds to a fixed point is the separatrix of a toroidally symmetric configuration with a poloidal divertor. Here the stable

and unstable manifolds coincide to form the separatrix. When a perturbation appears, the separatrix splits into the stable and unstable manifolds which no longer coincide, but intersect transversally infinitely many times. Close to the X-point in the direction from which the field lines approach it (the stable direction) the stable manifold is close to the unperturbed separatrix, but the unstable manifold wildly oscillates, creating lobes that become longer and narrower when the X-point is approached<sup>1</sup>. In the direction of field lines leaving the X-point (the unstable direction) we obtain a similar picture with the roles of the stable and unstable manifolds reversed. This complex structure is called a homoclinic or heteroclinic tangle. An important property of the invariant manifolds is that field lines can't cross them, because field lines can't intersect. Invariant manifolds thus act as boundaries for the field lines. Field lines originating in the hot plasma core are contained inside the invariant manifolds of the X-point and the only way they can reach the divertor targets is when the lobes of the (un)stable manifolds near the X-point intersect the target plates<sup>11</sup>. By tracing the intersection of the manifolds with the plates one obtains curves which delimit the region connected to the plasma core, characterized by mostly high connection length in the laminar plot. Those divertor footprints typically take the form of long spiralling bands, each band corresponding to the intersection of a protruding lobe of a stable or unstable manifold with the divertor.

The divertor footprints have a complicated inner structure and not all points inside the manifolds have high connection length. Some of them are connected to the opposite divertor plate after two poloidal turns by laminar flux tubes which do not penetrate deeply under the separatrix. The points with high connection lengths are the images of invariant manifolds of the X-points of the inner island chains<sup>11</sup>. This fine structure was studied in detail in<sup>12</sup>. Here we focus on the overall shape of the divertor footprints which is given by the invariant manifolds of the divertor X-point and is better experimentally accessible.

The length of the spiral can be characterized by the maximum value of  $s$  reached, i.e. the value  $s_{\text{tip}}$  at its tip. The difference  $\Delta s_{\text{max}}$  of  $s_{\text{tip}}$  and the separatrix value  $s = 1$  expresses the radial distance on the divertor plate between the footprint's tip and the unperturbed strike point, which lies at the intersection of the unperturbed separatrix with the divertor plate. The unstable manifold<sup>18</sup> is the footprint's boundary and so the footprint's tip is the point on the manifold which is the most distant from the unperturbed separatrix, the distance in terms of  $s$  being  $\Delta s_{\text{max}}$ . The value  $\Delta s_{\text{max}}$  thus quantifies the magnitude of the separatrix splitting.

To estimate the separation between the unperturbed separatrix and the unstable manifold we will follow two field lines – one in the unperturbed field, lying on the separatrix, and the other in the perturbed field, lying on the unstable manifold. They are parameterized by the toroidal angle  $\varphi$ . Let us choose them so that they

are initially (in the vicinity of the X-point which they approach asymptotically when followed backwards in  $\varphi$ ) close to each other. The parametric equations of the perturbed field line are

$$s = s'(\varphi) \quad (21)$$

$$\phi = \phi'(\varphi). \quad (22)$$

For the unperturbed field line they can be written explicitly using the definition of  $\phi$  and the fact that the unperturbed field line lies on the separatrix where  $s = 1$ :

$$s = s(\varphi) = 1 \quad (23)$$

$$\phi = \phi(\varphi) = \varphi - \phi_0 \quad (24)$$

where the constant  $\phi_0$  determines the toroidal phase: it is the toroidal angle of the point where the field line crosses the outboard midplane.

The rate of change of  $s$  along the perturbed field line is

$$\frac{ds'}{d\varphi} = \delta b^1(s'(\varphi), \phi'(\varphi), \varphi) \quad (25)$$

If the perturbed field line does not deviate significantly as a result of the perturbation, we may use a first-order perturbative approximation and evaluate  $b^1$  on the unperturbed field line:

$$\frac{ds'}{d\varphi} = \delta b^1(s = 1, \phi(\varphi), \varphi) \quad (26)$$

The deviation  $\Delta s$  of the perturbed field line from the unperturbed one after a full poloidal turn is given by the integral of (26):

$$\Delta s(\phi_0) = \int_{-\infty}^{\infty} \delta b^1(s = 1, \phi(\varphi) = \varphi - \phi_0, \varphi) d\varphi \quad (27)$$

or using  $\phi$  as the parameter:

$$\Delta s(\phi_0) = S(\phi_0) \equiv \int_{-\infty}^{\infty} \delta b^1(s = 1, \phi, \phi + \phi_0) d\phi. \quad (28)$$

Note that  $\Delta s(\phi_0)$  is a function of the toroidal phase  $\phi_0$  and may be zero: this happens when the unstable manifold intersects the unperturbed separatrix.

The function  $S$  defined by the integral (28) is closely related to the Melnikov function  $M$ :

$$M(\phi_0) = \int_{-\infty}^{\infty} \delta b^\psi(s = 1, \phi, \phi + \phi_0) d\phi = \frac{d\psi}{ds} S(\phi_0) \quad (29)$$

where  $\delta b^\psi(s = 1, \phi, \phi + \phi_0) = \frac{d\psi}{ds} b^1(s = 1, \phi, \phi + \phi_0)$  is the contravariant component of  $\delta B$  with respect to the  $\psi$  coordinate. The only difference between  $M$  and  $S$  is that  $S$  gives the change of  $s$  while  $M$  gives the change of  $\psi$ <sup>3</sup>.

If there is only one toroidal mode  $\tilde{b}_n^1(\phi)$  of the perturbation [cf. equation (15)], the function  $S$  can be replaced by a single complex number  $\tilde{S}_n(s=1)$ :

$$\begin{aligned} S(\phi_0) &= \int_{-\infty}^{\infty} \delta b^1(s=1, \phi, \phi + \phi_0) d\phi \\ &= \int_{-\infty}^{\infty} 2\Re \left\{ \exp[-in(\phi + \phi_0)] \tilde{b}_n^1(s=1, \phi) \right\} d\phi \\ &= 2\Re \left[ \exp(-in\phi_0) \int_{-\infty}^{\infty} \exp(-in\phi) \tilde{b}_n^1(s=1, \phi) d\phi \right] \\ &= 2\Re \left[ \exp(-in\phi_0) \tilde{S}_n(s=1) \right] \end{aligned} \quad (30)$$

$\tilde{S}_n(s=1)$  is defined by the equation (19), which also naturally extends the definition of  $\tilde{S}_n(s)$  to the domain  $s < 1$ . Analogously the value  $\tilde{M}_n(s) \equiv \frac{d\psi}{ds} \tilde{S}_n(s)$  can be used to express the Melnikov function  $M$  as a single number  $\tilde{M}_n(s=1)$ .

The value  $\Delta s_{\max}$  is the maximum deviation of the unstable manifold:  $\Delta s_{\max} = \max_{\phi_0} \Delta s(\phi_0)$ . Using  $\tilde{S}_n(s=1)$  it is expressed as

$$\Delta s_{\max} = 2|\tilde{S}_n(s=1)| \quad (31)$$

The island widths and the magnitude of separatrix splitting, and consequently the length of the divertor footprints, are given by a single function  $\tilde{S}_n(s)$ : the island widths by its values at  $s < 1$  and the magnitude of splitting by the value at  $s = 1$ .

It has been already known that island widths and the magnitude of splitting can be described by a single radial function<sup>2</sup>: the Poincaré-type integral  $R_n$  which is an integral of the modes of the perturbed poloidal flux  $H_1$  (cf. (A8)) instead of the perturbed field  $\tilde{b}_n^1$ . It can be shown that  $\tilde{M}_n = \epsilon n R_n$  so for a single toroidal mode of the perturbation our formalism of the Melnikov-like functions  $S$  or  $M$  is equivalent to the Poincaré-type integral approach. The reference 2 gives also other results expressed in terms of the function  $R_n$  such as the width of the stochastic layer and the field line diffusion coefficient which can be also simply reformulated using Melnikov-like functions.

#### IV. SPECIFIC FORM OF THE MODES OF A LOCALIZED LOW-FIELD SIDE PERTURBATION

In the previous sections we introduced the Melnikov-like function  $\tilde{S}_n$  and showed how it expresses both the sizes of the magnetic islands and the sizes of the divertor footprints. This is not sufficient to relate the sizes of the footprints to the sizes of the islands unless the radial dependence of  $\tilde{S}_n$  is known. In this section an approximate form of this dependence at the edge will be given for the special case of external magnetic perturbations imposed by coils located at the low-field side. The motivation for this case is the use of such coils as an ELM

control mechanism<sup>13</sup> where the coils are supposed to impact the edge region where the ELMs originate.

We will use a simplified model of the perturbed magnetic field where the perturbation is localized at the low field side where the field line pitch angle  $d\varphi/d\theta$  ( $\theta$  being the geometric poloidal angle) is assumed to be constant poloidally and radially. This is a realistic assumption for the edge region near the separatrix which is our region of interest. We will note the local pitch angle  $q_1$ :  $q_1 = d\varphi/d\theta = \text{const}$ . The variation of the safety factor is assumed to be caused only by the variation of the pitch angle in the regions where the perturbation is negligible: the high-field side and especially the X-point. This requires the perturbation coils to be placed sufficiently far from the X-point region.

Along a field line in the low field side region we have

$$\theta^* = o\varphi/q = oq_1\theta/q. \quad (32)$$

It follows that the  $\phi$  function has a simple dependence on  $\theta$  in this region:

$$\phi = q_1\theta. \quad (33)$$

The  $\bar{m}$  Fourier component of the perturbation w.r.t the geometric poloidal angle  $\theta$  is defined as

$$\tilde{b}_{(\bar{m},n)}^1 = \frac{1}{2\pi} \int \exp(-i\bar{m}\theta) \tilde{b}_n^1(\theta) d\theta \quad (34)$$

where  $\tilde{b}_n^1(\theta)$  is the  $n$  toroidal Fourier component of  $\delta b^1(\theta, \varphi)$  considered as a function of  $\theta$ :

$$\tilde{b}_n^1(\theta) \equiv \tilde{b}_n^1(\phi(\theta)). \quad (35)$$

We will now find the relation between the Fourier components  $\tilde{b}_{(m,n)}^1$  and  $\tilde{b}_{(\bar{m},n)}^1$ . We are neglecting the perturbation outside the region where the Eq. (33) holds which allows to express  $\tilde{b}_{(m,n)}^1$  [Eq. (16)] in terms of the  $\theta$  coordinate:

$$\tilde{b}_{(m,n)}^1 = \frac{q_1}{o2\pi q} \int_{-\pi}^{\pi} \exp\left[-io\left(\frac{mq_1}{q}\theta\right)\right] \tilde{b}_n^1(\theta) d\theta \quad (36)$$

Equations (36) and (35) finally give simple relations between  $\tilde{b}_{(m,n)}^1$  and  $\tilde{b}_{(\bar{m},n)}^1$ :

$$\tilde{b}_{(m,n)}^1 = \frac{q_1}{q} \tilde{b}_{(mq_1/q,n)}^1 = \frac{q_1}{q} \tilde{b}_{(nq_1,n)}^1 \quad (37)$$

and

$$\tilde{b}_{(\bar{m},n)}^1 = \frac{q}{q_1} \tilde{b}_{(\bar{m}q/q_1,n)}^1.$$

From those it can be seen why the maxima and minima of the spectrum  $\tilde{b}_{(m,n)}^1$  in  $(m, s)$  space form “ridges” and “valleys” aligned with the  $q$  profile, as can be seen e.g. for the proposed ITER designs in Ref. 7 (Fig. 15c therein) and noted for DIII-D in Ref. 14 (see Fig. 1b therein). The Fourier component w.r.t.  $\theta^* - \tilde{b}_{(m,n)}^1 -$

is given by Eq. (37). Assuming that the Fourier component of the perturbation w.r.t.  $\theta - \tilde{b}'^1_{(\bar{m},n)}$  does not change significantly between different magnetic surfaces, the only radial dependence is the inverse proportionality to  $q$  which is the same for all the poloidal modes. If  $\tilde{b}^1_{(m,n)}$  has the maximum on one surface with  $s = s_1$  for  $m = m_{\max}(s_1)$ , on other resonant surface with  $s = s_2$  it will have also maximum for  $m = m_{\max}(s_2)$  equal to  $m_{\max}(s_1)q(s_2)/q(s_1)$  so maxima will be aligned with the  $q$  profile which is given in the  $(m, s)$  space as the set of points satisfying  $m = nq(s)$ .

Using these results the an approximate radial dependence of  $\tilde{b}^1_{(\bar{m},n)}$  can be found. The radial dependence of the geometric poloidal Fourier component  $\tilde{B}^r_{\bar{m}}$  of the radial perturbation  $B^r \equiv \delta \vec{B} \cdot \vec{e}_r$  (with  $\vec{e}_r$  being the unit vector perpendicular to the magnetic surfaces) is<sup>15</sup>  $\tilde{B}^r_{\bar{m}} \propto r^{\bar{m}-1}$ . The contravariant  $s$  component  $B^1$  is given by  $B^1 = B^r \partial s / \partial r$ . Assuming that  $\partial s / \partial r$  and  $B^3$  do not depend significantly on the poloidal angle in the area with a non-negligible perturbation, the geometric poloidal Fourier component of the normalized contravariant perturbation is given by

$$\tilde{b}'^1_{(\bar{m},n)} \propto r^{\bar{m}-1} \frac{\partial s}{\partial r} \frac{1}{B^3}. \quad (38)$$

The radial dependence of  $\tilde{b}^1_{(m,n)}$  is given by the formula (37) where Eq. (38) can be used to substitute for  $\tilde{b}'^1_{(mq_1/q,n)}$ .

For resonant modes we may use the Melnikov-like function instead. From Eqs. (33), (18) and (34) it follows that

$$\tilde{S}_n(s) = \frac{q_1}{o2\pi} \tilde{b}'^1_{(nq_1,n)} \quad (39)$$

The radial dependence of  $\tilde{S}_n(s)$  can be obtained from Eqs. (38) and (39):

$$\tilde{S}_n(s) \propto q_1 r^{nq_1-1} \frac{\partial s}{\partial r} \frac{1}{B^3} \quad (40)$$

At a sufficiently narrow edge region the right-hand side is not strongly radially dependent, so we may expect the values of  $\tilde{S}_n(s)$  on different resonant surface to be strongly correlated. Note that (40) and this conclusion applies also to the value on separatrix  $\tilde{S}_n(s=1)$ , which is thus the limit of  $\tilde{S}_n(s)$  at the resonant surfaces approaching the separatrix, because (40) does not contain discontinuous terms.

## V. CONCLUSION

We derived a generalized formula for analytic estimation of width of magnetic islands which does not rely on a simplified cylindrical geometry, but instead takes into account toroidal toroidal geometry and arbitrary (i.e. non-circular) cross-section of magnetic surfaces. This makes

it especially suitable for estimating the edge ergodization in an X-point tokamak geometry, where the edge region is substantially different from a cylindrical approximation. The formula is based on the perturbed magnetic field and we demonstrated its equivalence to formulae expressed in terms of the perturbed poloidal flux. We then formulated assumptions about the form of the perturbed magnetic field which correspond to the perturbations typically used in the ongoing effort to control ELMs with magnetic perturbations on a range of tokamaks. Namely, we suppose that the perturbation acts mostly in a region away from the X-point, where the pitch angle of the field lines does not have a significant radial variation in the region of interest, which is the edge zone near the separatrix. This assumption is valid for the coils used for ELM control experiments in most tokamaks, as well as the proposed coils for ITER. Using this assumption we then derived more concrete results about the perturbation harmonics which determine the island sizes. We demonstrated that all the resonant harmonics are correlated. Our result expresses formally the alignment of the maxima and minima of the perturbation spectra with the safety factor profile, which is often observed in the calculations of perturbation harmonics. We also show that the quantity which determines the island sizes is also directly linked to the Melnikov integral and thus determines the extent of the footprints on the divertor plates.

Our results show that by using coils on low field side it is not possible to create significantly different resonant perturbations on different rational surfaces. Maximizing the resonant mode on one surface also leads to maximization of resonant modes on other surfaces. This is advantageous if one wants to optimize the coil system for maximum island overlap and stochastization. If one rather wants to study the effect on perturbation on each surface separately it might be more advantageous to choose a different position of the coils, as it is the case for the new perturbation coils on DIII-D. Maximizing the island overlap will also lead to maximization of divertor footprints due to the relation between island sizes and the Melnikov integral.

As our method is restricted to a LFS-localized perturbation, the results do not apply to a perturbation field created inside the plasma itself, e.g. a locked mode. In this case the relation between magnetic islands and the divertor footprints may be much less constrained.

## Acknowledgments

This work was partly supported by the European Communities under the contracts of association between EURATOM, IPP.CR and CEA. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

## Appendix A: Hamiltonian representation of field lines and magnetic islands

In the theory of hamiltonian dynamical systems (see Ref. 16), the formula for the island width is derived using the hamiltonian description of field line dynamics, with the poloidal flux function in the role of the hamiltonian and the toroidal angle in the role of time (see e.g. Ref. 2). The hamiltonian is defined as

$$H = A_\varphi = RA_{\hat{\varphi}} \quad (\text{A1})$$

where  $A_\varphi$  is the covariant toroidal component of the vector potential and  $A_{\hat{\varphi}} = \vec{\mathbf{A}} \cdot \hat{\mathbf{e}}_\varphi$  is the physical component, with  $\hat{\mathbf{e}}_\varphi$  being the unit basis vector in the toroidal direction. A convenient choice of canonical coordinates is the action-angle representation, where the action is the toroidal flux  $\Phi$  and the angle is  $\theta^*$ . The Hamiltonian equations are:

$$\frac{d\theta^*}{d\varphi} = \frac{\partial H}{\partial \Phi} \quad (\text{A2})$$

$$\frac{d\Phi}{d\varphi} = -\frac{\partial H}{\partial \theta^*} \quad (\text{A3})$$

In the equilibrium case  $H$  is a function of poloidal position only and is independent on the toroidal angle  $\varphi$ . Moreover,  $\Phi$  and  $\theta^*$  being action-angle variables, they are chosen so that  $H$  is only a function of  $\Phi$  and Eq. (A3) is identically zero. A nonaxisymmetric perturbation is represented by the addition of a small term  $\epsilon H_1(\Phi, \theta^*, \varphi)$  to the hamiltonian, which can then be written as

$$H(\Phi, \theta^*, \varphi) = H_0(\Phi) + \epsilon H_1(\Phi, \theta^*, \varphi). \quad (\text{A4})$$

$H_0$  is the equilibrium part, which can be identified with the unperturbed poloidal flux  $\psi$ . The perturbed part  $\epsilon H_1(\Phi, \theta^*, \varphi)$  corresponds to a perturbation  $\delta A_\varphi(\Phi, \theta^*, \varphi)$  of  $A_\varphi$ . The equilibrium part has the property

$$\frac{dH_0}{d\Phi} = \frac{d\psi}{d\Phi} = \frac{1}{q} \quad (\text{A5})$$

which reduces Eq. (A2) to the form (7). (We assume that the perturbation term  $\epsilon \frac{dH_1(\Phi, \theta^*, \varphi)}{d\Phi}$  is negligible in

comparison with the equilibrium term  $1/q$  and can be neglected.) The equation (6) can be derived from (A3) by expressing the perturbed field  $\delta b^1$  using the perturbed potential  $\delta A_\varphi$ . This expression is

$$\delta b^1 = -\frac{ds}{d\psi} \frac{1}{q} \frac{\partial \delta A_\varphi}{\partial \theta^*}. \quad (\text{A6})$$

The derivative  $\frac{ds}{d\varphi}$  can be expressed as

$$\frac{ds}{d\varphi} = \frac{ds}{d\psi} \frac{d\psi}{d\Phi} \frac{d\Phi}{d\varphi} = \frac{ds}{d\psi} \frac{1}{q} \frac{d\Phi}{d\varphi}. \quad (\text{A7})$$

From Eqs. (A6) and (A7) it follows that Eq. (6) is equivalent to (A3).

It is useful to decompose the perturbed potential in Fourier modes, analogously to the decomposition (1) of  $\delta b^1$ :

$$\delta A_\varphi = \epsilon H_1 = \epsilon \sum_{m,n=-\infty}^{\infty} \tilde{H}_{(m,n)} \exp[i(m\theta^* - n\varphi)] \quad (\text{A8})$$

$$= \epsilon \sum_{m,n} H_{(m,n)} \cos(m\theta^* - n\varphi + \chi_{mn}) \quad (\text{A9})$$

From (1), (A8) and (A6) we obtain the relation between  $\tilde{b}_{(m,n)}^1$  and  $\tilde{H}_{(m,n)}$ :

$$\tilde{b}_{(m,n)}^1 = -\frac{ds}{d\psi} \frac{1}{q} i m \epsilon \tilde{H}_{(m,n)} \quad (\text{A10})$$

$$b_{(m,n)}^1 = \frac{ds}{d\psi} \frac{1}{q} m \epsilon H_{(m,n)}. \quad (\text{A11})$$

The half-width of islands measured in terms of the action variable (toroidal flux  $\Phi$ ) is<sup>16</sup>:

$$\delta_\Phi = 2q \sqrt{\frac{\epsilon H_{(m,n)}}{\frac{dq}{d\Phi}}} \quad (\text{A12})$$

In a linear approximation, the half-width in terms of  $s$  is related to  $\delta_\Phi$  by the relation  $\delta_s = \frac{ds}{d\Phi} \delta_\Phi$ . Moreover,  $\frac{dq}{d\Phi} = \frac{ds}{d\Phi} q'$  and  $\frac{ds}{d\Phi} = \frac{ds}{d\psi} \frac{1}{q}$ , so using Eq. (A11) we see that the expressions (A12) and (13) are equivalent.

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